Introduction to Dynamical Systems

Solutions Problem Set 4

Exercise 1. Assume that ϕ, Φ are as in the structural stability theorem in lecture3.pdf. Also assume with

$$\Phi(x) = 2x + \hat{\psi}(x), \quad \hat{\psi}(x+1) = \hat{\psi}(x),$$

that $\hat{\psi} \in C^1(\mathbb{R})$, and finally that $\hat{\psi}(0) = 0$, $\hat{\psi}'(0) \neq 0$. Then show that there is no open interval $(a, b) \subset \mathbb{R}$ on which the function u which solves

$$\Phi(u(x)) = u(2x) \tag{1}$$

is of class C^1. *Hint: first show that if u is C^1 on some open interval, then it is already C^1 everywhere. Then show that there is a dense set of points where the derivative u' vanishes.*

Solution. Notice that u solving the fixed point problem (1) means that for $x \in (a, b)$ where u is differentiable, we have

$$u'(2x) = u'(x)\left(1 + \frac{\hat{\psi}'(u(x))}{2}\right). \tag{2}$$

Since $u \in C^1((a, b))$ and $\hat{\psi} \in C^1(\mathbb{R})$, we find that $u \in C^1((2a, 2b))$, and by iteration,

$$u \in C^1\left(\bigcup_{n\geq 0}(2^n a, 2^n b)\right).$$

Finally, notice that there must be $N \geq 0$ for which $2^N(b - a) > 1$, and this means that u is differentiable in $(2^N a, 2^N b)$, which contains an interval of length 1, and by periodicity u is differentiable everywhere.

To show that u' vanishes in a dense set, notice that since $\hat{\psi}$ is Lipschitz with $L \in [0, 1)$, the rightmost factor on the right-hand side of (2) is never equal to zero, and therefore

$$u'(x) = 0 \text{ if and only if } u'(2x) = 0. \tag{3}$$

Moreover, by looking at (2) at $x = 0$ one can use this same argument to show that $u'(0) = 0$. Differentiating $u(x + 1) = u(x) + 1$ implicitly yields

$$u'(x + 1) = u'(x),$$

meaning that $u'(n) = 0$ for all $n \in \mathbb{Z}$. Combining this with (3) and iterating $k \in \mathbb{N}$ times yields

$$u'\left(\frac{n}{2^k}\right) = 0.$$

Hence u' vanishes on the set $S = \left\{n/2^k : n \in \mathbb{Z},\, k \in \mathbb{N} \cup \{0\}\right\}$, which is dense in $\mathbb{R}$. Therefore $u' \equiv 0$, which contradicts the fact that u is increasing. $\square$

Exercise 2.

1. Let T be a contraction on the complete metric space (X, d) with contraction factor $\lambda \in (0, 1)$, i.e. $d(Tx, Ty) \leq \lambda d(x, y)$. Then if x^* denotes the fixed point of T, show that for any $x_1 \in X$ there is a constant $C > 0$ such that

$$d\left(T^k(x_1), x^*\right) \leq C\lambda^k.$$

2. Let $Tg(x) = \Phi^{-1}(g(2x))$ be as in lecture3.pdf, acting on the complete metric space X as in the lecture. By optimizing the choice of k in (i), show that there is a number $\alpha = \alpha(L) \in (0,1)$ and a constant $C_* > 0$ such that

$$|u(x) - u(y)| \leq C_* |x - y|^\alpha$$

for every $x, y \in \mathbb{R}$ with $|x - y| \leq 1$. Here u solves the fixed point problem (1). thus the conjugating function h is Hölder continuous.

Solution. For the first part, notice that

$$\begin{aligned} d(T^k x_1, x^*) &= \lim_{n\to\infty} d(T^k x_1, T^n x_1) \leq \lambda^k \sup_{n\geq 1} d(x_1, T^n x_1) \\ &\leq \lambda^k \sup_{n\geq 1} \sum_{j=1}^n d(T^{j-1}x_1, T^j x_1) \leq \lambda^k \sum_{j\geq 1} \lambda^j d(x_1, Tx_1) = \frac{1}{1-\lambda} d(x_1, Tx_1)\lambda^k. \end{aligned}$$

It suffices to set $C = d(x_1, Tx_1)/(1-\lambda)$. For the second part, coming back to the proof in the lectures, we notice that $T^k g(x) = \Phi^{-k}(2^k x)$, which implies

$$\left|T^k(g(x)) - T^k(g(y))\right| \leq 2^k r_1^{-k} \cdot |x - y| .$$

Using the first part of the exercise, we conclude that

$$|u(x) - u(y)| \leq 2d(u, T^k g) + \left|T^k(g(x)) - T^k(g(y))\right| \leq 2Cr_1^{-k} + 2^k r_1^{-k} \cdot |x - y| .$$

Now, let $\alpha > 0$ and choose k to be the smallest natural number such that

$$2^k \geq \frac{C}{|x-y|}, \quad 2^\alpha = r_1,$$

to find

$$|u(x) - u(y)| \leq 4 \cdot 2^{k-1} |x - y| r_1^{-k} \leq 4C \left(\frac{|x-y|}{C}\right)^\alpha \leq D |x - y|^\alpha .$$

Finally, since $r_1 = 2 - L$, notice that we may explicitly write

$$\alpha = \frac{\log(2-L)}{\log 2}.$$

□

Exercise 3. Let X a finite measure space and $T\colon X \longrightarrow X$ measure preserving. For measurable sets $A, B \subset X$, write

$$A \triangle B := (A \setminus B) \cup (B \setminus A) .$$

Show that T is ergodic if and only if

$$m(A \triangle T^{-1}(A)) = 0$$

implies that $m(A) = 0$ or $m(A) = m(X)$. *Hint:* $\bigcap_{n\geq 1} \bigcup_{j\geq n} T^{-j}(A)$.

Solution. Notice that the *only if* direction is direct. In particular, if $A \subset X$ is invariant by T, then the symmetric difference of A and $T^{-1}(A)$ is zero, and T is automatically ergodic.

For the *if* direction, let $A \subset X$ be such that the condition of the statement is satisfied. We let

$$A^+ = \bigcap_{n\geq 1} \bigcup_{j\geq n} T^{-j}(A).$$

Then, it is immediate that A^+ is invariant by T, and

$$m(T^{-j}(A) \triangle T^{-(j+1)(A)}) = m(T^{-j}(A \triangle T^{-1}(A))) = m(A \triangle T^{-1}(A)) = 0,$$

which readily implies that for any $k, \ell \in \mathbb{N}$,

$$
\begin{aligned}
m(T^{-k}(A) \triangle T^{-(k+\ell)}(A)) &= m(T^{-k}(A \triangle T^{-\ell}(A))) \\
&= m(A \triangle T^{-\ell}(A)) \leq \sum_{j=0}^{\ell-1} m(T^{-j}(A) \triangle T^{-(j+1)}(A)) = 0.
\end{aligned}
$$

In particular, we achieve

$$
T^{-k}(A) \cup T^{-(k+\ell)}(A) = T^{-k}(A) \sqcup S_k, \quad m(S_{k\ell}) = 0.
$$

Collecting all the preimages,

$$
A^+ = \bigcap_{n \geq 1} \bigcup_{j \geq n} T^{-j}(A) = \bigcap_{n \geq 1} \left[T^{-n}(A) \sqcup \bigcup_{\ell \geq 1} S_{n\ell} \right] = N \sqcup \bigcap_{n \geq 1} T^{-n}(A),
$$

where $m(N) = 0$. Through the continuity of the measure and thanks to a simple induction argument we conclude that

$$
m\left(\bigcap_{j \geq 1} T^{-j}(A) \right) = m(A),
$$

and therefore $m(A) = m(A^+)$. Hence either $m(A) = m(X)$ or $m(A) = 0$, thanks to the ergodicity of T. □

Exercise 4. Let (X, m) be a finite measure space and $T\colon X \longrightarrow X$ a measure preserving map. Using the Von Neumann Mean Ergodic Theorem, show that if $f \in L^1(X, m)$, then

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j(x))
$$

exists in $L^1(X, m)$, and is a T-invariant function.

Solution. We begin by taking $A > 0$ and letting

$$
f_A(x) = f(x) \cdot \mathbb{1}_{\{|f| \leq A\}}(x) \in L^2(X, m).
$$

Then, notice that $f_A \in L^2(X, m)$ and f_A converges to f in $L^1(X, m)$ as $A \to \infty$,

$$
\|f - f_A\|_{L^1} = \left\| f \cdot \mathbb{1}_{\{|f| > A\}} \right\|_{L^1} \underset{A \to \infty}{\longrightarrow} 0.
$$

We choose $\varepsilon > 0$ and let $A > 0$ large enough so that $\|f - f_A\|_{L^1} < \varepsilon/3$, and when we apply the theorem to $f_A \in L^2(X, m)$ we find that

$$
\frac{1}{N} \sum_{j=0}^{N-1} f_A \circ T^j \underset{N \to \infty}{\longrightarrow} f_A^*,
$$

in the sense of $L^2(X, m)$, where f_A^* in invariant by T. Since this converges, it is in particular a Cauchy sequence, and we can estimate

$$
\left\| \frac{1}{N} \sum_{j=0}^{N-1} f_A \circ T^j - \frac{1}{M} \sum_{j=0}^{M-1} f_A \circ T^j \right\|_{L^1} \leq \left\| \frac{1}{N} \sum_{j=0}^{N-1} f_A \circ T^j - \frac{1}{M} \sum_{j=0}^{M-1} f_A \circ T^j \right\|_{L^2} \sqrt{m(X)} < \varepsilon/3
$$

for all N, M sufficiently big, using the Cauchy-Schwarz inequality together with the fact that $m(X) < \infty$. Now, using the triangle inequality we find

$$
\left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^j \right\|_{L^1} < \varepsilon,
$$

if N, M are sufficiently large. Therefore $\frac{1}{N}\sum_{j=0}^{N-1} f \circ T^j$ is a Cauchy sequence in $L^1(X, m)$. Since this is a Banach space, there is $f^* \in L^1(X, m)$ such that $\frac{1}{N}\sum_{j=0}^{N-1} f \circ T^j \to f^*$. It remains to check that f^* is invariant under T. To this end, notice that

$$\begin{aligned} \|f^* \circ T - f^*\|_{L^1} &= \lim_{N\to\infty} \left\| \frac{f \circ T^N - f}{N} \right\|_{L^1} \\ &\leq \lim_{N\to\infty} \frac{\left\|f \circ T^N\right\|_{L^1} + \|f\|_{L^1}}{N} = \lim_{N\to\infty} \frac{2\|f\|_{L^1}}{N} = 0. \end{aligned}$$

□